Games, Logic, and Constructive Sets

Edited by
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Part I

Logic and Games
Logic and Game Theory: Close Encounters of the Third Kind

Johan van Benthem

This paper is an exercise at the interface of logic and game theory. We show how games may be analyzed in the style of logical process theories, starting from the pure action case with perfect information, and then including more realistic features like imperfect information and preferences.

1 Encounters of three kinds

Games are a species of processes with intricate dynamic interactions between players about which we have vivid first-hand intuitions. Historically, this richness has led to several mathematizations, starting with the probabilistic one of the 17th century, emphasizing odds and outcomes in betting games. In the 20th century, economic game theory developed a powerful account of strategic games of various kinds, and equilibria between strategies. But there is still a third level of interest: the fine-structure of players' actions, deliberations, and decisions as they move along — and here contemporary logic is beginning to meet game theory. There are several strands to this contact. Logicians have long employed logic games for analyzing argumentation, semantic evaluation, model comparison, or model construction. Winning strategies in these games capture central logical notions such as proof, truth, or similarity. These are very specialized games from a game-theoretical viewpoint, lacking finer utilities for players, and assuming perfect information about any play. But there is also logical structure to the general games in the game-theoretic literature. The first, and most

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established trend at the interface comes from philosophical logic. Reasoning by players in the course of a game, or by an observer making sense of the game, naturally involves notions like knowledge, belief, and belief revision. This is the area of epistemic logic, conditional logic, and belief revision theory — and there has even been an independent rediscovery of epistemic logic inside the game theory of the 1970s. Much of the interest here focuses on a better understanding of rationality of actors in a complex setting. The second, less established trend comes from computer science. It looks at games as distributed processes involving intelligent agents, an interactive setting that is typical for modern internet-based computational tasks, from information processing to electronic commerce. Tools for analyzing game structure in this sense include dynamic logic, temporal logic, and theories of concurrency like process algebra. One major concern here is the shifting balance of expressive power and algorithmic complexity in complex interactive processes.

This paper is a short introduction to these interfaces in the process perspective with a sprinkling of epistemic concerns. We refer to other papers for proofs of results: our interest here is the over-all setting. Moreover, our discussion presupposes some acquaintance with the essentials of modal logic and game theory (cf. Blackburn et al. (2001); Osborne and Rubinstein (1994)).

2 Games from a logical viewpoint

2.1 Pictures

To most people, ‘game theory’ conjures up pictures like the 2x2 matrices for Prisoner’s Dilemma — or less worn-out examples like the ‘Stag Hunt’, where two players must choose one of two available strategies independently:

<table>
<thead>
<tr>
<th></th>
<th>hunt stag</th>
<th>hunt rabbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>hunt stag</td>
<td>3,3</td>
<td>0,1</td>
</tr>
<tr>
<td>hunt rabbit</td>
<td>1,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

The story is this. The game has two ‘equilibria’: one where both players hunt stag (an arduous enterprise requiring cooperation, yielding a high pay-off for each), and one where both hunt rabbit (individual, less arduous, but also less rewarding). Choosing different strategies is not a stable form of joint behaviour: a stag hunter confronted with a rabbit hunter is better off switching to rabbit hunting. More generally, a pair
of strategies is in *Nash equilibrium* if neither player can improve her outcome by deviating when the other player sticks to his strategy.

Other typical game-theoretic pictures are 'Centipedes', such as the game depicted here

\[ A \quad E \quad A \quad E \quad 5,3 \]

\[ 1,0 \quad 0,2 \quad 3,1 \quad 2,4 \]

with values for outcomes indicated in the displayed pairs of (A-value, E-value). Here the story is as follows. Players can move 'down' or 'right'. Arguing backward from the right, at each state of the game, the active player is best off going down, leading to a prediction that the game will stop after A's first move. This is surprising, because the far-right outcome (5, 3) seems certainly better for both. The Centipede has generated much controversy about 'rational behaviour' — but we just display it for its structure. As opposed to the Stag Hunt, a global *strategic game*, it is an extensive game displaying all local moves. Moreover, it has *perfect information*: players always know what the other has done so far. In the Stag Hunt, however, players choose their actions in ignorance of what the other did: a form of *imperfect information*. Now, what does all this have to do with logic?

### 2.2 Games as models for logical languages

**Actions in modal and dynamic logic** The most obvious foothold for logical analysis is in extensive games. Consider the following 'game form' for two players A, E, involving states (nodes in the tree), moves (labeled arrows), runs (maximal sequences of successive moves), and outcome states for the runs:

![Game Tree](figure1)

Figure 1

Just as it stands, this tree is a model for a *modal logic* of actions (reachability via players' moves) with further special structure encoded in local
properties indicating turns at intermediate nodes, and other relevant properties \( p \) at nodes. One reads this picture with a global dynamic intuition. The game involves players’ strategies, assigning an available move to every node where it is their turn. Playing a strategy guarantees an outcome in the set of end states arising from every possible counterplay by the other player. This is the associated power for forcing the game into a specified set of outcomes. In this particular game,

- **A** has 2 strategies, with powers \( \{x, y\}, \{z, u\} \)
- **E** has 4 strategies, with powers \( \{x, z\}, \{x, u\}, \{y, z\}, \{y, u\} \)

Strategic powers can be described in a dynamic logic allowing complex actions. E.g., with \( \cup \) standing for choice between moves, the following box-diamond formula says that player **E** has a strategy forcing a set of outcomes satisfying \( p \):

\[
(a \cup b)(c \cup d)p
\]

**Preference logics** Now consider the preceding game with players’ preferences added, in the form of utility values for both players at outcome states:

![Figure 2](image1.png)

A utility-maximizing agent **E** will play strategy (left: \( d \), right: \( c \)), and taking this into account, a rational agent **A** plays move \( a \). Richer models like this interpret preference logics of various sorts, such as conditional logic, or logics of belief. E.g., we may naturally say that players believe the game will end in \( y \) — at least assuming that everyone plays rationally, so that ‘preference engenders plausibility’.

**Imperfect information and epistemic logic** Finally, consider an imperfect information version of our game, where **E** does not know which move was played by **A**. This ignorance may arise for several reasons, as in real games — either private or public. Perhaps **E** did not pay attention to **A**’s move, or the latter was hidden from her (as
with initial deals in a game of cards). Her uncertainty is graphically indicated by the dotted line in the following picture:

![Figure 3](image)

This picture suggests extending our language to an epistemic logic with knowledge operators, expressing things at E’s turn like

\[
K_E(\langle a \rangle p \lor \langle b \rangle p) \quad \text{E knows she has a move guaranteeing p}
\]

\[
\neg K_E(a)p \land \neg K_E(b)p \quad \text{but she does not know which one does.}
\]

### 2.3 Intensional logic as usual

Logical languages of these various kinds express assertions about players’ actions and their effects, which can be checked against game models in the usual semantical way. In addition to this model-checking mode, one can study proof systems axiomatizing valid principles for reasoning in special game classes, satisfying constraints such as unique-valuedness of moves, ‘rationality’ of behaviour under preferences, or ‘perfect recall’ in imperfect information games. This business-as-usual makes game logic into a rich form of applied intensional logic (cf. Stalnaker (1999)).

In this paper, however, we wish to develop a number of new themes that arise on the process view of games. For convenience, extensive games will mostly be taken to be finite trees.

### 2.4 Major issues in process theories

Existing logical process theories show a wide variety, depending on one’s viewpoint on the relevant dynamic structure. Here are some key features (van Benthem (1996); Harel et al. (2000)).

a) At one extreme, processes are black boxes with external input / output only — at another, one cares for all decisions and actions that go into their internal states. One measures such grain levels in terms of ‘simulations’ between process graphs. If all decisions are relevant, some form of bisimulation is needed — with only
external behaviour in focus, one uses less discriminating notions of so-called trace equivalence.

b) The other side of the coin are internal languages for describing a process: modal ones if all actions are relevant, much coarser ones if only traces matter.

c) Instead of working inside, one can look from the outside, putting together whole processes by natural operations such as choice or sequential composition. This takes external languages of operations, as in process algebra (Fokkink (2000)).

d) Finally, processes come in two flavours. Finite processes correspond to bounded tasks, described by terminating programs. But natural infinite processes should run forever, such as operating systems. The logical setting for the latter is some form of temporal logic, or extensions of dynamic logic like the ‘μ-calculus’.

All these issues apply immediately to games. As good logicians, we start by assuming perfect information, lifting this restriction later.

3 Levels of representation

3.1 Games: from actions to outcomes

In setting one’s sights, there are various natural levels of representation for a game, ranging from local actions to global outcomes. Do you play soccer for the brilliant moves, or just for the score-board? The difference may be high-lighted by means of a ‘propositional game’. The propositional Distribution law \( p \land (q \lor r) \leftrightarrow (p \land q) \lor (p \land r) \) suggests a comparison between two 2-person games with choices for players A and E:

\[ \text{Are the following two games the same?} \]

![Figure 4](image-url)
The intuitive answer is. "Yes, qua players' outcome control, No, qua intermediate actions." Clearly, the two games differ qua action structure. E.g., only on the left, but not on the right, E can be in a position to make the game end in both y and z. But if one analyzes available strategies, players' powers are indeed the same in both games depicted:

for E: \{x, y\}, \{x, z\} corresponding to her strategies "left" and "right"
for A: \{x\}, \{y, z\} for his strategies "left", "right" (on the left), and \langle "left", "left" \rangle, \langle "right", "right" \rangle (on the right)

Note A also has powers \{x, z\} and \{y, x\} in the right-hand tree, corresponding to his strategies \langle "left", "right" \rangle and \langle "right", "left" \rangle — but these are just supersets of those given, and hence weaker powers.

The outcome level corresponds roughly to the earlier 'strategic form' of a game, but with the full enumeration of all strategies suppressed. (In the preceding example, the full strategy table is a 2 \times 4 matrix.) The action level corresponds to the 'extensive form'. Later on, we shall also propose natural intermediate levels.

Now, as stated before, there are two styles of analysis for game representation. We can either define a simulation (a 'structural invariance'), or design a description language for the relevant game-internal properties. And these two styles can be matched precisely, as is well-known from logical process theories. The paradigmatic example is the match between 'bisimulation' of two processes, and their having the same properties definable in an internal modal language describing transitions. We can take this 'two-faced' style of analysis to games.

3.2 Actions in modal and dynamic logic

Expressive power of the language Extensive games were models for a modal or dynamic language which states, amongst others, 'strategic assertions'. In the game of Figure 1, player E had a strategy forcing outcome p:

\[ [a \cup b] (c \cup d) p \]

Thus, modal logic provides a systematic way of reasoning about players in games. A more general analysis of positions where E has such a strategy requires a modal fixed-point formalism like the \( \mu \)-calculus. Let the assertion \{E\}p state that

player E has a strategy for playing the game from now on which forces a set of outcomes all satisfying p.